DIGITAL FILTERS

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## INTRODUCTION

This report analyzes Butterworth filters in the time domain by using a digital method for finding their transient performances. Digital filter compensation is effected by multiplication in the Z-transform domain by $(1+z^m)/2$. For the sake of comparison, posicast compensation (6) of a compensator $\left[ (1-P)+Pz^m \right]$ is employed in each corresponding digital filter compensation case. The analytical digital process used in this report is an approximate method of solving differential equations, the Naumov program (4), which is an improved accuracy trapezoidal convolution program (3).

Previous work on Butterworth filters, pseudo-Butterworth filters, posicast control and Naumov program are reviewed.

Practical examples of analyzing and improving the performances of Butterworth filters from second order to fourth order with unit step inputs are presented.

It is necessary to note that the posicast compensation method results in larger undershoots and time delay when the order of the Butterworth filter being operated becomes higher. These defects are absent in digital filter compensation. Detailed observations are presented in the Discussion section.

## BUTTERWORTH FUNCTIONS

A transfer function with spectrum

$$G_n(j\omega) = \frac{1}{\sqrt{1+\omega^{2n}}} \qquad (1)$$

is the $n^{th}$ order Butterworth filter or maximally flat low-pass response and it is an approximation to the ideal low-pass filter. The nature of the approximate function is seen from two observations:

(1) From the binomial series expansion

$$(1+X)^{-n} = \sum_{k=0}^{\infty} \binom{n-k+1}{k} (-X)^k \qquad X^2 < 1 \qquad (2)$$

it is seen that the Butterworth response when $\omega$ is near zero is

$$(1+\omega^{2n})^{-1/2} = 1-(1/2)\omega^{2n} + (3/8)\omega^{4n} - (5/16)\omega^{6n} + \dots \quad (3)$$

and that the first $(2n-1)$ derivatives are zero at $\omega = 0$.

(2) $\left| G_n(j\omega) \right|$ has fixed points ( $\left| G_{n+1} \right| = \left| G_n \right|$ for all n) at $\omega = 0$, $\omega = 1$ and $\omega = \infty$ for all integers n. Corresponding values for $\left| G_n(j\omega) \right|$ are 1, 0.707, and 0.

The Butterworth transfer functions are now determined. The poles of this function are defined by the equation,

$$1 + (-s^2)^n = 0$$

Then the pole locations are

$$s_k = e^{j(2k-1)\pi/2n} \qquad \qquad \text{n even}$$
$$s_k = e^{j(2k)\pi/2n} \qquad \qquad \text{n odd}$$
or $\quad s_k = e^{j(2k+n-1)\pi/2n} \qquad \qquad k = 1,2,3,\dots\dots,2n$

The poles so defined are evenly spaced on the unit circle in the s-plane and have symmetry with respect to both the real and the imaginary axes. For n odd, a pair of poles are located on the real axis, but the poles are not located on the imaginary axis for either n even or n odd. These properties accrue because the poles are separated by $\pi/n$ radians, and are located $\pi/2n$ radians from the real axis for n even and on the real axis for n odd.

Straightforward calculation of

$$G_n(s) = \frac{1}{a_0 + a_1 s + a_2 s^2 + a_3 s^3 + \ldots + a_n s^n}$$

$$= \frac{1}{\prod_{k=0}^{n/2} (s-s_k)(s-\bar{s}_k)} \tag{4}$$

yields the coefficients $a_k$ of the denominator polynomial of $G_n(s)$. These Butterworth polynomial coefficients are given in Table 1 and were found in reference (7). Here, the definition

$$G_n(s) \equiv 1/B_n(s) \tag{4a}$$

is made.

## PSEUDO-BUTTERWORTH FUNCTIONS

The pseudo-Butterworth function was first introduced by Marcel J. E. Golay (2) and developed by K. A. Pullen, Jr. Pullen (5) extracted the basic polynomials from Golay's paper and worked out the coefficient relations.

The pseudo-Butterworth polynomial of degree n is

$$B_p(s) = b_0 + b_1 s + b_2 s^2 + \ldots + b_{n-1} s^{n-1} + b_n s^n \tag{5}$$

$$= \sum_{k=0}^{n} b_k s^k \tag{6}$$

This polynomial can be approximately converted into a true Butterworth type by replacing s with $s/(b_n)^{1/n}$. The approximate Butterworth polynomial is

$$B_n'(s) \equiv \sum_{k=0}^{n} a_k s^k \tag{7}$$

where $\qquad a_k = b_k/(b_n)^{k/n} \tag{8}$

Table 1.  Coefficients of Butterworth Polynomials

$$B_n(s) = \prod_{k=0}^{n/2} (s - s^k)(s - \bar{s}_k)$$

| n\k | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1.0 | | | | | | | | |
| 1 | 1.0 | 1.0000 | | | | | | | |
| 2 | 1.0 | 1.4142 | 1.0000 | | | | | | |
| 3 | 1.0 | 2.0000 | 2.0000 | 1.0000 | | | | | |
| 4 | 1.0 | 2.6131 | 3.4142 | 2.6131 | 1.0000 | | | | |
| 5 | 1.0 | 3.2361 | 5.2361 | 5.2361 | 3.2361 | 1.0000 | | | |
| 6 | 1.0 | 3.8637 | 7.4641 | 9.1416 | 7.4641 | 3.8637 | 1.0000 | | |
| 7 | 1.0 | 4.4940 | 10.0978 | 14.5918 | 14.5918 | 10.0978 | 4.4940 | 1.0000 | |
| 8 | 1.0 | 5.1258 | 13.1371 | 21.8462 | 25.6884 | 21.8462 | 13.1371 | 5.1258 | 1.0000 |

The coefficients of the first ten orders of pseudo-Butterworth polynomial and their corresponding normalized coefficients of the first eight orders of approximate Butterworth polynomial are presented in Table 2 and Table 3 respectively.

The basic coefficient recurrence relation of an $n^{th}$ order pseudo-Butterworth polynomial is

$$c_j^n = (n+j-2)\, c_{j-1}^{n-1} + c_j^{n-1} \qquad 0 \leq j \leq n \qquad (9)$$

where $\qquad n \geq 2$

The coefficients of zero and first order are

$$c_0^0 = 1$$

$$c_0^1 = 1, \qquad c_1^1 = 1$$

Note that the superscripts are not powers.

Table 2. Coefficients of Approximate Butterworth Polynomials

$$B'_n(s) = \sum_{k=0}^{n} s_k s^k \ , \ a_k = b_k/(b_n)^{k/n}$$

When $b_k$'s are the coefficients of pseudo-Butterworth polynomial of order n.

| n \ k | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-------|---|---|---|---|---|---|---|---|---|
| 0 | 1.0 | | | | | | | | |
| 1 | 1.0 | 1.0000 | | | | | | | |
| 2 | 1.0 | 1.4142 | 1.0000 | | | | | | |
| 3 | 1.0 | 2.0000 | 2.0000 | 1.0000 | | | | | |
| 4 | 1.0 | 2.6596 | 3.4687 | 2.6392 | 1.0000 | | | | |
| 5 | 1.0 | 3.3454 | 5.4833 | 5.4388 | 3.3684 | 1.0000 | | | |
| 6 | 1.0 | 4.0438 | 8.0134 | 9.9837 | 7.8102 | 4.0000 | 1.0000 | | |
| 7 | 1.0 | 4.7662 | 11.0079 | 16.0660 | 15.9415 | 10.7663 | 4.6125 | 1.0000 | |
| 8 | 1.0 | 5.4694 | 14.7004 | 25.2384 | 29.6392 | 24.8341 | 14.4298 | 5.4718 | 1.0000 |

The additional relations are

$$C_0^n = 1$$
$$C_1^n = (1/2)(n)^{(2)} + 1$$
$$C_2^n = (1/8)(n+1)^{(4)} + (1/2)(n+1)^{(2)} + 1$$
$$C_3^n = (1/48)(n+2)^{(6)} + (1/8)(n+2)^{(4)} - (1/2)(n+2)^{(2)} + 3$$
$$C_4^n = (1/384)(n+3)^{(8)} + (1/48)(n+3)^{(6)} - (1/8)(n+3)^{(4)} + (3/2)(n+3)^{(2)} - 15$$
$$C_5^n = (1/3840)(n+4)^{(10)} + (1/384)(n+4)^{(8)} + (1/48)(n+4)^{(6)} + (3/8)(n+4)^{(4)}$$
$$- (15/2)(n+4)^{(2)} + 105$$

etc.

where $n^{(k)} \equiv n(n-1)(n-2)\ldots\ldots(n-k+1)$

Table 3. Coefficients of Pseudo-Butterworth Polynomials

$$C_j^n = (n+j-2)\,C_{j-1}^{n-1} + C_j^{n-1}, \quad 0 \leq j \geq n, \quad n \geq 2$$

| n \ j | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | | | | | |
| 1 | 1 | 1 | | | | | | | | | |
| 2 | 1 | 2 | 2 | | | | | | | | |
| 3 | 1 | 4 | 8 | 8 | | | | | | | |
| 4 | 1 | 7 | 24 | 48 | 48 | | | | | | |
| 5 | 1 | 11 | 59 | 192 | 384 | 384 | | | | | |
| 6 | 1 | 16 | 125 | 605 | 1920 | 3840 | 3840 | | | | |
| 7 | 1 | 22 | 237 | 1605 | 7365 | 23040 | 46080 | 46080 | | | |
| 8 | 1 | 29 | 413 | 3738 | 23415 | 104055 | 322560 | 645120 | 645120 | | |
| 9 | 1 | 37 | 674 | 7868 | 64533 | 385035 | 1675275 | 5160960 | 10321920 | 10321920 | |
| 10 | 1 | 46 | 1044 | 15282 | 158949 | 1223964 | 7065765 | 31290085 | 92897280 | 185794560 | 185794560 |

The closed form for $C_j^n$ is

$$C_j^n = \frac{(n+j-1)^{(2j)}}{2^j \cdot j!} + \frac{(n+j-1)^{(2j-2)}}{2^{j-1} \cdot (j-1)!} + \sum_{u=2}^{n} \frac{(-1)^{u-1}(2u-3)(2u-5)\ldots 5 \cdot 3 \cdot 1(n+j-1)^{(2n-2u)}}{2^{n-u} \cdot (n-u)!} \quad (10)$$

and the pseudo-Butterworth polynomial can be written as

$$B_p(s) = \sum_{j=0}^{n} C_j^n s^j \qquad (11)$$

## POSICAST CONTROLS

The "posicast" or "positive-cast" control method was first introduced by Otto J. M. Smith (6). This method can be applied only when the damping of the complex poles is small. The secret of this method is the division of the input function into two times and two amplitudes. The second input time must follow the first input time by exactly one-half cycle of the transient if the superposition of the results is to be zero for all subsequent times. Any other input-spacing time would excite two oscillations whose vectors would not cancel, because they would always have some phase-angle difference. The first quantity at the first input time must have the value that can bring the maximum overshoot to reach the desired final output value at the second input time. The second quantity must have the value which can maintain that peak value produced by the first quantity.

Figure 1 shows a natural damped transient with amplitude A at one peak and amplitude B at the following peak. The total motion is A+B. Therefore a step A+B should be broken into A initially and B later. The control mechanism should have the transfer function as

$$P_c = \frac{A + Be^{-sT_n/2}}{A + B} \qquad (12)$$

The principle of vector cancellation with posicast control can also be expressed by Fig. 1. In Fig. 1,

(a) Decrement of a damped oscillation. Vector A which starts the oscillation decreases to magnitude B after time $T_{n/2}$. (one-half cycle later).

(b) Required excitation function. Vector B is instantaneously equal to vector A and $180^\circ$ out of phase with it.

(c) Response of the oscillatory system when driven by the excitation function in Fig. 1(b), is

$$A \cdot u(t) + B \cdot u(t - T_{n/2})$$

the Laplace transform of which is $(A + Be^{-sT_n/2})/s$.

## THE NAUMOV PROGRAM

Naumov (4) solved differential equations in the time domain employing trapezoidal integration on a multiple integration formula,

$$\frac{1}{n!} \int_0^t (t-\tau)^n f(\tau) d\tau \qquad (13)$$

A look-up table for $t^n/n!$ for various times $t$ and exponents n is placed in computer storage. A method is then given for employing the look-up table in conjunction with the differential equation to obtain the solution.

Recently Criswell (1) devised an algorithm that easily computes exact $Z(1/s^n)$. The algorithm readily generates tables of Z-transforms of $t^n/n!$ for n much larger than those in presently available tables. Such a large table of Z-transforms can be used for checking and improving approximate
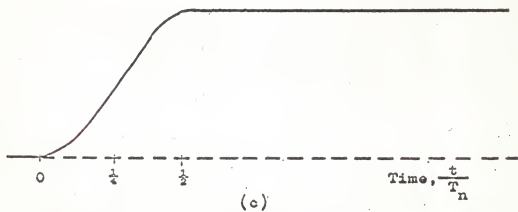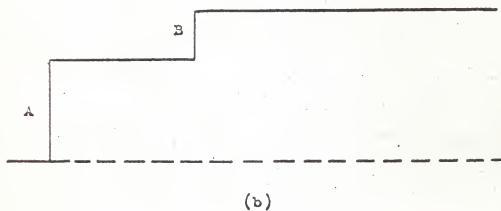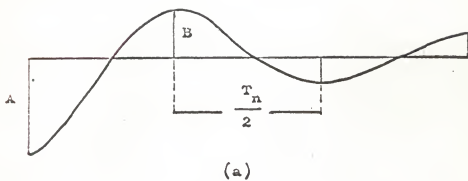
Fig.1. Principle of vector cancellation with Posicast control

integration formulas generated by trapezoidal convolution (3). Replacement of approximate Z-transforms of $1/s^n$ by their exact Z-transforms yields a trapezoidal convolution integrator substitution program identical to Naumov's method but dispensing with digital computer memory for storing look-up table of $t^n/n!$ required in Naumov's method.

A brief derivation of Criswell's algorithm is now presented. Since it is known that

$$Z(1/s^n) = \left[ Tz/(n-1) \right]\left[\frac{d}{dz} Z(1/s^{n-1})\right], \; z \equiv e^{-Ts} \qquad (14)$$

then the first few calculations of $Z(1/s^n)$ can be easily carried out. These calculations yield

$$Z(1/s) = \frac{T^0 z}{1} \cdot \frac{z^{-1}}{1-z} \qquad (15a)$$

$$Z(1/s^2) = \frac{Tz}{1} \cdot \frac{1}{(1-z)^2} \qquad (15b)$$

$$Z(1/s^3) = \frac{T^2 z}{2} \cdot \frac{(1 + z)}{(1 - z)^3} \qquad (15c)$$

$$Z(1/s^4) = \frac{T^3 z}{6} \cdot \frac{z^2 + 4z + 1}{(1 - z)^4} \qquad (15d)$$

Assume that the previous results can be written for any integer n in the canonical form

$$Z(1/s^n) = \frac{T^{n-1} z}{(n-1)!} \; \frac{A_n(z)}{(1-z)^n} \qquad (16)$$

Where $A_n(z)$ is a polynomial of degree n-2. Differentiation of equation (16) shows that $Z(1/s^{n+1})$ is also in the canonical form. Hence by mathematical

induction the canonical form holds for any positive n. And, the differential-recurrence relation which can generate the desired algorithm is

$$A_n(z) = (1-z) \frac{d}{dz} (zA_{n-1}) + (n-1)zA_{n-1} \text{ for } n \geq 2 \qquad (17)$$

Recall that the starting polynomial is $A_1(z) = z^{-1}$.

Define $A(n,p)$ to be the coefficient of $z^p$ in $A_n(z)$. The differential-recurrence relation for $A_n(z)$ induces the coefficient relation

$$A(n,p) = (p + 1)A(n-1,p) + (n-p-1)A(n-1,p-1) \qquad (18)$$

The coefficients are presented in Table 4 with their corresponding n's and p's.

## DIGITAL FILTERS

Conversion of a linear continuous filter function into digital form can be accomplished by the Naumov program in the Z-transform domain. The linear continuous filter functions are Butterworth functions. Since the Naumov program gives exact $Z(1/s^n)$, the transient response of a digital filter can approach the exact response of the Butterworth filter by choosing the smaller sampling interval sizes, T. The numerical calculations are carried out by a digital computer. Conversion of the Butterworth filter into a digital filter with unit step function input is shown for the second, third and fourth order Butterworth filters.

### Second Order Digital Butterworth Filter

If the second order Butterworth filter is

$$G_2(s) = \frac{1}{1 + \sqrt{2}s + s^2} \qquad (19)$$

# Table 4

COEFFICIENTS $\mathcal{A}(\alpha, p)$ IN $A_n(x) = \sum_{p=-1}^{n-2} A(\alpha, p)\, x^p$

FOR $Z(1/s^n) = \frac{I^{n-1}x}{(\alpha-1)!}\cdot\frac{A_n(x)}{(1-x)^n}$

| $n$ \ $p$ | -1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | | | | | | | | | | | | | | |
| 2 | | 1 | | | | | | | | | | | | | |
| 3 | | 1 | 1 | | | | | | | | | | | | |
| 4 | | 1 | 4 | 1 | | | | | | | | | | | |
| 5 | | 1 | 11 | 11 | 1 | | | | | | | | | | |
| 6 | | 1 | 26 | 66 | 26 | 1 | | | | | | | | | |
| 7 | | 1 | 57 | 302 | 302 | 57 | 1 | | | | | | | | |
| 8 | | 1 | 120 | 1191 | 2416 | 1191 | 120 | 1 | | | | | | | |
| 9 | | 1 | 247 | 4293 | 15619 | 15619 | 4293 | 247 | 1 | | | | | | |
| 10 | | 1 | 502 | 14,608 | 88,234 | 156,190 | 88234 | 14608 | 502 | 1 | | | | | |
| 11 | | 1 | 1013 | 47,840 | 455,192 | 1,310,354 | 1,310,354 | 455192 | 47840 | 1013 | 1 | | | | |
| 12 | | 1 | 2036 | 152,637 | 2,203,488 | 9,738,114 | 15,724,248 | 9738114 | 2203488 | 152637 | 2036 | 1 | | | |
| 13 | | 1 | 4083 | 478,271 | 10,187,685 | 66,318,474 | 162,512,286 | 162512286 | 66318474 | 10187685 | 478271 | 4083 | 1 | | |
| 14 | | 1 | 8178 | 1479726 | 55,533,450 | 423281535 | 1505621508 | 2275172004 | 1505621508 | 423281535 | 45533450 | 1479726 | 8178 | 1 | |
| 15 | | 1 | 16369 | 4537314 | 193410786 | 2571742175 | 1284326363 | 27971176092 | 27971176092 | 12843262363 | 2571742175 | 123410736 | 4537314 | 16369 | 1 |

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then the output function with unit step input is

$$\bar{y} = \frac{1}{s} \frac{1}{1 + \sqrt{2}s + s^2} \tag{20}$$

$$y(0) = \dot{y}(0) = 0$$

Rearranging this equation and dividing by $s^2$ gives

$$(1 + \frac{\sqrt{2}}{s} + \frac{1}{s^2})\bar{y} = \frac{1}{s^3} \tag{21}$$

Taking the Z-transform of both sides yields

$$Z\left[(1 + \frac{2}{s} + \frac{1}{s^2})\bar{y}\right] = Z(\frac{1}{s^3}) \tag{22}$$

Then, by employing both the Naumov program and the approximate Z-transform (3) of convolution, $(\frac{1}{s^n}\bar{y})$ gives

$$Z(\frac{\sqrt{2}}{s}\bar{y}) = \frac{\sqrt{2}(1+z)}{2(1-z)} Z\bar{y} \tag{23a}$$

$$Z(\frac{1}{s^2}\bar{y}) = \frac{T^2z}{(1-z)^2} Z\bar{y} \tag{23b}$$

$$Z(\frac{1}{s^3}) = \frac{T^2z(1+z)}{2(1-z)^3} \tag{23c}$$

Substituting equation (23) into equation (22) and then multiplying by $(1-z)^2$ yields

$$\left[z^2(1 - \frac{\sqrt{2}}{2}) + z(T^2 - 2) + (\frac{\sqrt{2}}{2}T + 1)\right] Z\bar{y}$$

$$= \frac{T^2z(1+z)}{2(1-z)} \tag{24}$$

The corresponding recurrence relation is

$$y_n = \frac{2(2-T^2)}{\sqrt{2}T + 2} y_{n-1} + \frac{\sqrt{2}T-2}{\sqrt{2}T+2} y_{n-2} + \frac{T^2 x_n}{\sqrt{2}T+2} \tag{25}$$

$$x_n = \left\{ 0, \ 1, \ 2, \ 2, \ 2, \ 2, \ 2, \ \cdots\cdots\cdots, \ 2, \ \cdots\cdots \right\}$$

Let $\quad A = 2(2 - T^2)$

$\qquad B = \sqrt{2}T - 2$

$\qquad C = T^2$

$\qquad D = \sqrt{2}T - 2$

Then $\quad y_n = \left[ (Ay_{n-1} + By_{n-2})/D \right] + (C/D)x_n \tag{26}$

When  $n=0$ $\qquad\qquad y_0 = 0;$

$\qquad n=1$ $\qquad\qquad y_1 = C/D;$

$\qquad n \geqslant 2$ $\qquad\qquad y_n = (Ay_{n-1} + By_{n-2} + 2C)/D.$

If the sampling interval T is chosen as 0.1 second, then the graphical and numerical solutions of y(t) are shown in Fig. 2 and Table 5 respectively. It is noted that the graphical solution is based on numerical calculations.

### Third Order Digital Butterworth Filter

The third order Butterworth filter is

$$B_3(s) = \frac{1}{1 + 2s + 2s^2 + s^3} \tag{27}$$

Following the procedures and assumptions of the second order case yields

$$Z \left[ (\frac{1}{s^3} + \frac{2}{s^2} + \frac{2}{s} + 1)\bar{y} \right] = Z(\frac{1}{s^4}) \tag{28}$$

A: $y(t) = \mathcal{L}^{-1}\left(\dfrac{1/s}{s^2 + 2s + 1}\right)$

B: $y(t) = \mathcal{L}^{-1}\left(\dfrac{1/s}{s^3 + 2s^2 + 2s + 1}\right)$

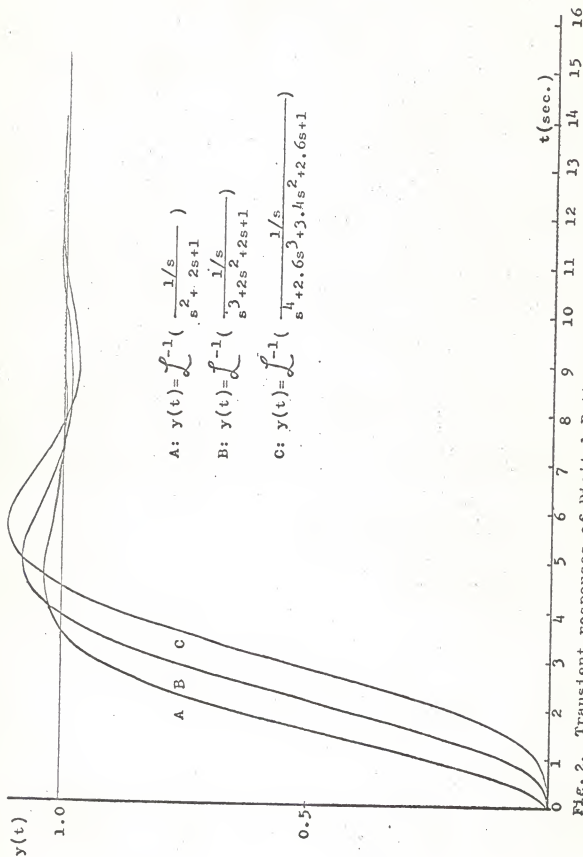C: $y(t) = \mathcal{L}^{-1}\left(\dfrac{1/s}{s^4 + 2.6s^3 + 3.4s^2 + 2.6s + 1}\right)$

Fig. 2. Transient responses of Digital Butterworth filters with unit step input.

Table 5. $y(t) = \mathcal{L}^{-1}\left( \dfrac{1/s}{s^2+\sqrt{2}s + 1} \right)$, $y(0) = 0$, $T = 0.1$

| t | y(t) | t | y(t) | t | y(t) |
|---|------|---|------|---|------|
| 0.0 | 0.0000 | 2.7 | 0.9096 | 5.4 | 1.0307 |
| 0.1 | 0.0017 | 2.8 | 0.9385 | 5.5 | 1.0288 |
| 0.2 | 0.0180 | 2.9 | 0.9455 | 5.6 | 1.0268 |
| 0.3 | 0.0338 | 3.0 | 0.9607 | 5.7 | 1.0248 |
| 0.4 | 0.0685 | 3.1 | 0.9743 | 5.8 | 1.0229 |
| 0.5 | 0.0979 | 3.2 | 0.9864 | 5.9 | 1.0210 |
| 0.6 | 0.1343 | 3.3 | 0.9970 | 6.0 | 1.0192 |
| 0.7 | 0.1739 | 3.4 | 1.0062 | 6.1 | 1.0174 |
| 0.8 | 0.2160 | 3.5 | 1.0141 | 6.2 | 1.0157 |
| 0.9 | 0.2598 | 3.6 | 1.0209 | 6.3 | 1.0141 |
| 1.0 | 0.3048 | 3.7 | 1.0266 | 6.4 | 1.0126 |
| 1.1 | 0.3504 | 3.8 | 1.0313 | 6.5 | 1.0111 |
| 1.2 | 0.3959 | 3.9 | 1.0350 | 6.6 | 1.0097 |
| 1.3 | 0.4411 | 4.0 | 1.0380 | 6.7 | 1.0085 |
| 1.4 | 0.4856 | 4.1 | 1.0402 | 6.8 | 1.0073 |
| 1.5 | 0.5290 | 4.2 | 1.0417 | 6.9 | 1.0062 |
| 1.6 | 0.5710 | 4.3 | 1.0426 | 7.0 | 1.0052 |
| 1.7 | 0.6116 | 4.4 | 1.0431 | 7.1 | 1.0042 |
| 1.8 | 0.6513 | 4.5 | 1.0430 | 7.2 | 1.0034 |
| 1.9 | 0.6873 | 4.6 | 1.0426 | 7.3 | 1.0026 |
| 2.0 | 0.7222 | 4.7 | 1.0418 | 7.4 | 1.0019 |
| 2.1 | 0.7552 | 4.8 | 1.0408 | 7.5 | 1.0013 |
| 2.2 | 0.7861 | 4.9 | 1.0394 | 7.6 | 1.0008 |
| 2.3 | 0.8149 | 5.0 | 1.0379 | 7.7 | 1.0003 |
| 2.4 | 0.8416 | 5.1 | 1.0363 | 7.8 | 0.9999 |
| 2.5 | 0.8663 | 5.2 | 1.0345 | 7.9 | 0.9995 |
| 2.6 | 0.8889 | 5.3 | 1.0326 | | |

The Naumov program gives

$$Z(\frac{2}{s}\bar{y}) = \frac{(1+z)T}{(1-z)}Z\bar{y} \tag{29a}$$

$$Z(\frac{2}{s^2}\bar{y}) = \frac{2T^2z}{(1-z)^2}Z\bar{y} \tag{29b}$$

$$Z(\frac{1}{s^3}\bar{y}) = \frac{T^3z(1+z)}{2(1-z)^3}Z\bar{y} \tag{29c}$$

$$Z(\frac{1}{s^4}) = \frac{T^3z}{6}\frac{z^2+4z+1}{(1-z)^4} \tag{29d}$$

Then

$$Z\bar{y}\left[(T-1)z^3+(3-T-2T^2+0.5T^3)z^2+(0.5T^3+2T^2-T-3)z+(T+1)\right]$$

$$= \frac{T^3}{6}\frac{z(z^2+4z+1)}{(1-z)} \tag{30}$$

Let

$A = (3+T-2T^2-0.5T^3)$

$B = (T+2T^2-0.5T^3-3)$

$C = 1-T$

$D = T^3$

$R = 1+T$

Then the recurrence relation is

$$y_n = \left[(Ay_{n-1}+By_{n-2}+Cy_{n-3})/R\right] + (\frac{D}{6R})x_n \tag{31}$$

$$x = \left\{0,\ 1,\ 5,\ 6,\ 6,\ 6,\ \ldots\ldots\ 6,\ \ldots\right\}$$

When

$n = 0 \qquad y_0 = 0;$

$n = 1 \qquad y_1 = \frac{D}{6R};$

$n = 2 \qquad y_2 = \frac{A}{R}y_1 + \frac{5D}{6R};$

$$n \geq 3 \qquad y_n = (Ay_{n-1} + By_{n-2} + Cy_{n-3}D)/R.$$

Results are presented in Fig. 2 and Table 6.

### Fourth Order Digital Butterworth Filter

The fourth order Butterworth filter is

$$G_4(s) = \frac{1}{s^4 + 2.6s^3 + 3.4s^2 + 2.6s + 1} \tag{32}$$

Calculation yields

$$Z\left[\left(\frac{1}{s^4} + \frac{2.6}{s^3} + \frac{3.4}{s^2} + \frac{2.6}{s} + 1\right)\bar{y}\right] = Z(\frac{1}{s^5}) \tag{33}$$

The Naumov program gives

$$Z(\frac{2.6}{s^2}\bar{y}) = \frac{1.3T(1+z)}{(1-z)}z\bar{y} \tag{34a}$$

$$Z(\frac{3.4}{s^3}\bar{y}) = \frac{3.4T^2z}{(1-z)^2}z\bar{y} \tag{34b}$$

$$Z(\frac{2.6}{s^3}\bar{y}) = \frac{1.3T^3z(1+z)}{(1-z)^3}z\bar{y} \tag{34c}$$

$$Z(\frac{1}{s^4}\bar{y}) = \frac{T^4z(z^2+4z+1)}{6(1-z)^4}z\bar{y} \tag{34d}$$

$$Z(\frac{1}{s^5}) = \frac{T^4z(z^3+11z^2+11z+1)}{24(1-z)^5} \tag{34e}$$

Let $\quad W = \frac{T^4}{6}$

$\qquad E = \frac{T^4}{24}$

Table 6. $y(t) = \mathcal{L}^{-1}\left(\dfrac{1}{s(s^3+2s^2+2s+1)}\right)$, $y(0)=0$, $T=0.1$ sec.

| t | y(t) | t | y(t) | t | y(t) |
|---|------|---|------|---|------|
| 0.0 | 0.0000 | 3.2 | 0.8750 | 6.4 | 1.0300 |
| 0.1 | 0.0002 | 3.3 | 0.9012 | 6.5 | 1.0256 |
| 0.2 | 0.0012 | 3.4 | 0.9254 | 6.6 | 1.0214 |
| 0.3 | 0.0038 | 3.5 | 0.9477 | 6.7 | 1.0174 |
| 0.4 | 0.0086 | 3.6 | 0.9681 | 6.8 | 1.0137 |
| 0.5 | 0.0160 | 3.7 | 0.9866 | 6.9 | 1.0101 |
| 0.6 | 0.0263 | 3.8 | 1.0033 | 7.0 | 1.0067 |
| 0.7 | 0.0397 | 3.9 | 1.0181 | 7.1 | 1.0036 |
| 0.8 | 0.0562 | 4.0 | 1.0311 | 7.2 | 1.0008 |
| 0.9 | 0.0758 | 4.1 | 1.0424 | 7.3 | 0.9982 |
| 1.0 | 0.0985 | 4.2 | 1.0521 | 7.4 | 0.9959 |
| 1.1 | 0.1242 | 4.3 | 1.0602 | 7.5 | 0.9938 |
| 1.2 | 0.1525 | 4.4 | 1.0668 | 7.6 | 0.9920 |
| 1.3 | 0.1834 | 4.5 | 1.0721 | 7.7 | 0.9904 |
| 1.4 | 0.2165 | 4.6 | 1.0760 | 7.8 | 0.9891 |
| 1.5 | 0.2515 | 4.7 | 1.0788 | 7.9 | 0.9890 |
| 1.6 | 0.2882 | 4.8 | 1.0805 | 8.0 | 0.9871 |
| 1.7 | 0.3263 | 4.9 | 1.0811 | 8.1 | 0.9864 |
| 1.8 | 0.3654 | 5.0 | 1.0809 | 8.2 | 0.9859 |
| 1.9 | 0.4053 | 5.1 | 1.0798 | 8.3 | 0.9856 |
| 2.0 | 0.4465 | 5.2 | 1.0781 | 8.4 | 0.9854 |
| 2.1 | 0.4861 | 5.3 | 1.0757 | 8.5 | 0.9854 |
| 2.2 | 0.5264 | 5.4 | 1.0727 | 8.6 | 0.9856 |
| 2.3 | 0.5663 | 5.5 | 1.0693 | 8.7 | 0.9858 |
| 2.4 | 0.6056 | 5.6 | 1.0656 | 8.8 | 0.9862 |
| 2.5 | 0.6441 | 5.7 | 1.0615 | 8.9 | 0.9867 |
| 2.6 | 0.6815 | 5.8 | 1.0572 | 9.0 | 0.9872 |
| 2.7 | 0.7176 | 5.9 | 1.0528 | 9.1 | 0.9878 |
| 2.8 | 0.7524 | 6.0 | 1.0482 | 9.2 | 0.9885 |
| 2.9 | 0.7856 | 6.1 | 1.0436 | 9.3 | 0.9892 |
| 3.0 | 0.8172 | 6.2 | 1.0390 | 9.4 | 0.9900 |
| 3.1 | 0.8470 | 6.3 | 1.0344 | 9.5 | 0.9907 |

Substituting equation (34) into equation (33) yields

$$z\overline{y}\Big[(1+1.3T)+z(-4-2.6T+3.4T^2+1.3T^3+W) + z^2(6-6.8T^2+4W) +$$
$$z^3(-4+2.6T+3.4T^2-1.3T^3+W) + z^4(1-1.3T)\Big]$$
$$= \frac{Ez(z^3+11z^2+11z+1)}{(1-z)} \tag{35}$$

Let  $A = 4+2.6T-3.4T^2-1.3T^3-W$

$B = 6.8T^2-4W-6$

$C = 4-2.6T-3.4T^2+1.3T^3-W$

$D = 1.3T-1$

$F = 1+1.3T$

Then

$$y_n = \frac{(Ay_{n-1}+By_{n-2}+Cy_{n-3}+Dy_{n-4})}{F} + \left(\frac{Ex_n}{F}\right) \tag{36}$$

$$x_n = \Big\{0,\ 1,\ 12,\ 23,\ 24,\ 24,\ 24,\ \ldots\ldots\ 24,\ \ldots\ldots\Big\}$$

When  $n = 0$   $y_0 = 0;$

$n = 1$   $y_1 = \frac{E}{F};$

$n = 2$   $y_2 = (Ay_1+12E)/F;$

$n = 3$   $y_3 = (Ay_2+By_1+23E)/F;$

$n \geqslant 4$   $y_n = (Ay_{n-1}+By_{n-2}+Cy_{n-3}+Dy_{n-4}+24E)/F.$

Results are shown both in Table 7 and Fig. 2.

Table 7.  $y(t) = \mathcal{L}^{-1}\left(\dfrac{1/s}{s^4+2.6s^3+3.4s^2+2.6s+1}\right)$, $y(0)=0$, T=0.1 sec.

| t | y(t) | t | y(t) | t | y(t) |
|---|---|---|---|---|---|
| 0.0 | 0.0000 | 4.5 | 1.0226 | 9.0 | 0.9714 |
| 0.1 | 0.0000 | 4.6 | 1.0394 | 9.1 | 0.9717 |
| 0.2 | 0.0000 | 4.7 | 1.0543 | 9.2 | 0.9723 |
| 0.3 | 0.0003 | 4.8 | 1.0673 | 9.3 | 0.9732 |
| 0.4 | 0.0009 | 4.9 | 1.0785 | 9.4 | 0.9742 |
| 0.5 | 0.0020 | 5.0 | 1.0880 | 9.5 | 0.9754 |
| 0.6 | 0.0039 | 5.1 | 1.0957 | 9.6 | 0.9768 |
| 0.7 | 0.0068 | 5.2 | 1.1019 | 9.7 | 0.9784 |
| 0.8 | 0.0111 | 5.3 | 1.1064 | 9.8 | 0.9800 |
| 0.9 | 0.0168 | 5.4 | 1.1096 | 9.9 | 0.9817 |
| 1.0 | 0.0242 | 5.5 | 1.1113 | 10.0 | 0.9835 |
| 1.1 | 0.0334 | 5.6 | 1.1118 | 10.1 | 0.9854 |
| 1.2 | 0.0447 | 5.7 | 1.1111 | 10.2 | 0.9872 |
| 1.3 | 0.0581 | 5.8 | 1.1093 | 10.3 | 0.9891 |
| 1.4 | 0.0738 | 5.9 | 1.1066 | 10.4 | 0.9909 |
| 1.5 | 0.0917 | 6.0 | 1.1030 | 10.5 | 0.9927 |
| 1.6 | 0.1118 | 6.1 | 1.0987 | 10.6 | 0.9954 |
| 1.7 | 0.1342 | 6.2 | 1.0937 | 10.7 | 0.9962 |
| 1.8 | 0.1588 | 6.3 | 1.0882 | 10.8 | 0.9978 |
| 1.9 | 0.1855 | 6.4 | 1.0822 | 10.9 | 0.9994 |
| 2.0 | 0.2142 | 6.5 | 1.0759 | 11.0 | 1.0008 |
| 2.1 | 0.2447 | 6.6 | 1.0693 | 11.1 | 1.0022 |
| 2.2 | 0.2770 | 6.7 | 1.0626 | 11.2 | 1.0034 |
| 2.3 | 0.3107 | 6.8 | 1.0557 | 11.3 | 1.0045 |
| 2.4 | 0.3457 | 6.9 | 1.0488 | 11.4 | 1.0056 |
| 2.5 | 0.3818 | 7.0 | 1.0420 | 11.5 | 1.0065 |
| 2.6 | 0.4189 | 7.1 | 1.0353 | 11.6 | 1.0073 |
| 2.7 | 0.4565 | 7.2 | 1.0287 | 11.7 | 1.0079 |
| 2.8 | 0.4946 | 7.3 | 1.0224 | 11.8 | 1.0085 |
| 2.9 | 0.5330 | 7.4 | 1.0163 | 11.9 | 1.0090 |
| 3.0 | 0.5713 | 7.5 | 1.0106 | 12.0 | 1.0093 |
| 3.1 | 0.6093 | 7.6 | 1.0052 | 12.1 | 1.0095 |
| 3.2 | 0.6469 | 7.7 | 1.0001 | 12.2 | 1.0097 |
| 3.3 | 0.6839 | 7.8 | 0.9955 | 12.3 | 1.0098 |
| 3.4 | 0.7200 | 7.9 | 0.9912 | 12.4 | 1.0097 |
| 3.5 | 0.7551 | 8.0 | 0.9874 | 12.5 | 1.0096 |
| 3.6 | 0.7891 | 8.1 | 0.9840 | 12.6 | 1.0094 |
| 3.7 | 0.8217 | 8.2 | 0.9810 | 12.7 | 1.0092 |
| 3.8 | 0.8528 | 8.3 | 0.9785 | 12.8 | 1.0089 |
| 3.9 | 0.8824 | 8.4 | 0.9763 | 12.9 | 1.0085 |
| 4.0 | 0.9103 | 8.5 | 0.9746 | 13.0 | 1.0081 |
| 4.1 | 0.9365 | 8.6 | 0.9732 | 13.1 | 1.0077 |
| 4.2 | 0.9608 | 8.7 | 0.9723 | 13.2 | 1.0072 |
| 4.3 | 0.9833 | 8.8 | 0.9717 | 13.3 | 1.0068 |
| 4.4 | 1.0039 | 8.9 | 0.9714 | 13.4 | 1.0063 |

## DIGITAL FILTER COMPENSATION

Posicast control compensation can only be applied when the damping is small and introduces large time delay in the transient response; as the order of the system becomes higher, posicast compensation always results in large undershoots in the transient response. For the sake of reducing these defects, "digital filter compensation" is employed.
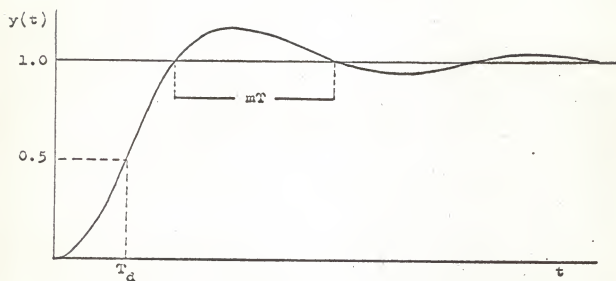
The basic idea of digital filter compensation is conversion the unit step function input into the double half-step function displayed in Fig. 3(b). The second half-step input follows the first half-step function after a time mT seconds, where mT is the time duration of the first overshoot of the Butterworth response to a unit step function. This is clarified in Fig. 3(a).

The symbol T is the duration between sampling times, and m is a constant to be determined for each Butterworth filter. Therefore a pre-compensating operater
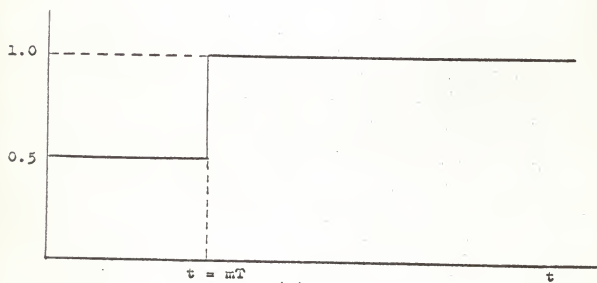
$$\frac{1 - e^{-msT}}{2} = \frac{1 - z^m}{2} \tag{37}$$

can be constructed. The block diagram of the system with this compensator is shown in Fig. 7(a).

Where X, Y and G(s) are the input, output and transfer (Butterworth filter) functions respectively. This completes the first step of compensation. This step reduces most of the overshoot of the original transient response but as happens in posicast control compensation, there now exists larger time delay which is much greater than that in the original system. Hence, the next step is to reduce time delay.

(a)

(b)

Fig. 3. Principle of double step input function of digital
filter compensation

Assume that the delay time in the original system is $T_{do}$ and after first step compensation is $T_{d1}$. Since $T_{d1}$ is greater than $T_{do}$, the ratio $T_{do}/T_{d1}$ is some value less than 1. The procedure is to replace s by $(T_{do}/T_{d1})$s in the compensated Butterworth polynomial. This replacement of s will renormalize time delay since $T_{do}/T_{d1}$ is less than 1, this meets the second requirement; so, the goal of compensation is accomplished.

Transient behaviors of several orders of Butterworth filter by applying digital filter compensation will be investigated in the next section.

### Compensation of Second Order Digital Butterworth Filter

A. First step of compensation

The time duration mT of the first overshoot of $G_2(s)$ with unit step input should be known; this can be found from Table 5 where

$$T = 0.1 \text{ sec.}$$

$$mT \doteq 7.8-3.4 = 4.4 \text{ sec.}$$

Therefore m = 44.

Then the system response with unit step input is shown in Fig. 7(b).

Thus

$$\bar{y} = (\frac{1}{s}) \ (\frac{1 + z^{44}}{2})(\frac{1}{s^2 + \sqrt{2}s + 1}) \tag{38}$$

Rearranging equation (38) by leaving $(\frac{1}{s})(\frac{1 + z^{44}}{2})$ at one side yields

$$(s^2 + \sqrt{2}s + 1)\bar{y} = (\frac{1}{s})\frac{1 + z^{44}}{2} \tag{39}$$

Taking the Z-transform yields after division by $s^2$ at both sides

$$Z \ \ (1 + \frac{\sqrt{2}}{s} + \frac{1}{s^2}) \ \bar{y} = Z \ \ \frac{1}{s^3} \ (\frac{1 + z^{44}}{2}) \tag{40}$$

After applying the Naumov program this equation yields

$$z\bar{y} \left[ (1 - \frac{\sqrt{2}}{2}T)z^2 + (T^2-2)z + (1 + \frac{\sqrt{2}}{2}T) \right]$$

$$= \frac{T^2 z(1+z)(1+z^{44})}{4(1-z)} \tag{41}$$

Let
$$A = 2(2-T^2)$$
$$B = \sqrt{2}T-2$$
$$C = T^2$$
$$D = \sqrt{2}T+2$$

Then the recurrence relation is

$$y_n = \frac{(Ay_{n-1}+By_{n-2})}{D} + (\frac{C}{2D})x_n \tag{42}$$

$$x_n = 0, 1, 2, \ldots 2, 3, 4, 4, \ldots 4, \ldots$$

This equation yields

$$y_0 = 0 \qquad\qquad n = 0$$

$$y_1 = \frac{C}{2D} \qquad\qquad n = 1$$

$$y_n = (Ay_{n-1}+By_{n-2}+C)/D \qquad\qquad 2 \leqslant n \leqslant 44$$

$$y_{45} = (Ay_{44}+By_{43}+1.5C)/D \qquad\qquad n = 45$$

$$y_n = (Ay_{n-1}+By_{n-2}+2C)/D \qquad\qquad n \geqslant 46$$

The numerical and graphical results of $y(t)$ are presented in Table 8 and Fig. 4 respectively.

B. The second step of compensation

From Table 5 and Table 8, the delay times which are found before and after the first step compensation are

$$T_{do} = 1.43 \text{ sec.}$$
$$T_{dl} = 3.35 \text{ sec.}$$

Table 8. $y(t)=\mathcal{L}^{-1}(\frac{1+z^{44}}{2}, \frac{1/s}{s^2+2s+1})$, $y(0)=0$, T=0.1 sec.

| t | y(t) | t | y(t) | t | y(t) |
|---|------|---|------|---|------|
| 0.0 | 0.0000 | 4.2 | 0.5209 | 8.5 | 1.0192 |
| 0.1 | 0.0023 | 4.3 | 0.5213 | 8.6 | 1.0200 |
| 0.2 | 0.0090 | 4.4 | 0.5215 | 8.7 | 1.0204 |
| 0.3 | 0.0194 | 4.5 | 0.5239 | 8.8 | 1.0206 |
| 0.4 | 0.0329 | 4.6 | 0.5303 | 8.9 | 1.0206 |
| 0.5 | 0.0490 | 4.7 | 0.5403 | 9.0 | 1.0204 |
| 0.6 | 0.0671 | 4.8 | 0.5533 | 9.1 | 1.0200 |
| 0.7 | 0.0896 | 4.9 | 0.5687 | 9.2 | 1.0195 |
| 0.8 | 0.1080 | 5.0 | 0.5861 | 9.3 | 1.0189 |
| 0.9 | 0.1299 | 5.1 | 0.6051 | 9.4 | 1.0181 |
| 1.0 | 0.1524 | 5.2 | 0.6252 | 9.5 | 1.0173 |
| 1.1 | 0.1752 | 5.3 | 0.6462 | 9.6 | 1.0165 |
| 1.2 | 0.1980 | 5.4 | 0.6678 | 9.7 | 1.0156 |
| 1.3 | 0.2206 | 5.5 | 0.6896 | 9.8 | 1.0147 |
| 1.4 | 0.2428 | 5.6 | 0.7114 | 9.9 | 1.0137 |
| 1.5 | 0.2645 | 5.7 | 0.7330 | 10.0 | 1.0128 |
| 1.6 | 0.2855 | 5.8 | 0.7543 | 10.1 | 1.0119 |
| 1.7 | 0.3056 | 5.9 | 0.7750 | 10.2 | 1.0109 |
| 1.8 | 0.3252 | 6.0 | 0.7951 | 10.3 | 1.0100 |
| 1.9 | 0.3436 | 6.1 | 0.8145 | 10.4 | 1.0092 |
| 2.0 | 0.3611 | 6.2 | 0.8330 | 10.5 | 1.0083 |
| 2.1 | 0.3776 | 6.3 | 0.8507 | 10.6 | 1.0075 |
| 2.2 | 0.3930 | 6.4 | 0.8674 | 10.7 | 1.0067 |
| 2.3 | 0.4074 | 6.5 | 0.8831 | 10.8 | 1.0060 |
| 2.4 | 0.4208 | 6.6 | 0.8979 | 10.9 | 1.0053 |
| 2.5 | 0.4331 | 6.7 | 0.9117 | 11.0 | 1.0046 |
| 2.6 | 0.4445 | 6.8 | 0.9244 | 11.1 | 1.0040 |
| 2.7 | 0.4548 | 6.9 | 0.9362 | 11.2 | 1.0035 |
| 2.8 | 0.4642 | 7.0 | 0.9470 | 11.3 | 1.0029 |
| 2.9 | 0.4727 | 7.1 | 0.9569 | 11.4 | 1.0025 |
| 3.0 | 0.4804 | 7.2 | 0.9659 | 11.5 | 1.0020 |
| 3.1 | 0.4872 | 7.3 | 0.9740 | 11.6 | 1.0016 |
| 3.2 | 0.4932 | 7.4 | 0.9813 | 11.7 | 1.0013 |
| 3.3 | 0.4985 | 7.5 | 0.9878 | 11.8 | 1.0009 |
| 3.4 | 0.5031 | 7.6 | 0.9936 | 11.9 | 1.0006 |
| 3.5 | 0.5071 | 7.7 | 0.9986 | 12.0 | 1.0004 |
| 3.6 | 0.5105 | 7.8 | 1.0030 | 12.1 | 1.0001 |
| 3.7 | 0.5133 | 7.9 | 1.0068 | 12.2 | 0.9999 |
| 3.8 | 0.5156 | 8.0 | 1.0101 | 12.3 | 0.9998 |
| 3.9 | 0.5175 | 8.1 | 1.0128 | 12.4 | 0.9996 |
| 4.0 | 0.5190 | 8.2 | 1.0150 | 12.5 | 0.9995 |
| 4.1 | 0.5201 | 8.3 | 1.0168 | | |

A: Transient response before compensation.
B: Transient response after first step of compensation.
C: Transient response after second step of compensation.
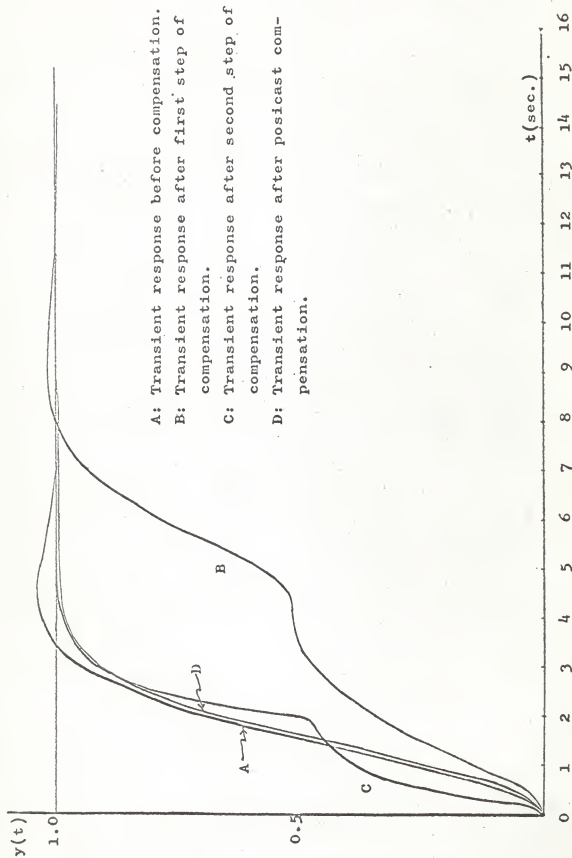D: Transient response after posicast compensation.

Fig. 4. Compensation of second order digital Butterworth filter with unit step function input.

Then

$$\frac{T_{do}}{T_{dl}} = \frac{1.43}{3.35} = 0.427$$

Replacing s by 0.427s in the transfer function $(\frac{1 + z^{44}}{2})(\frac{1}{1 + \sqrt{2}s + s^2})$
of equation (38) yields

$$\bar{y} = (\frac{1}{s})(\frac{1 + z^{19}}{2})(\frac{1}{1 + 0.605 s + 0.0181 s^2}) \tag{43}$$

Rearranging and taking the Z-transform of both sides yields

$$Z\left[(\frac{1}{s^2} + \frac{0.605}{s} + 0.0181)\bar{y}\right] = Z\left[\frac{1}{s^3}(\frac{1 + z^{19}}{2})\right] \tag{44}$$

Application of the Naumov program yields

$$Z\bar{y}\left[(0.0181 + 0.303T) + (T^2 - 0.0362)z + (0.0181 - 0.303T)z^2\right]$$

$$= \frac{T^2 z(1+z)(1+z^{19})}{4(1-z)} \tag{45}$$

Let
$$A = 0.0362 - T^2$$
$$B = 0.303T - 0.0181$$
$$C = T^2$$
$$D = 0.303T + 0.0181$$

Then, the recurrence relation is

$$y_n = \frac{(Ay_{n-1} + By_{n-1})}{D} + (\frac{0.25C}{D})x_n \tag{46}$$

$$x_n = \left\{0, 1, 2, 2, \ldots\ldots 2, 3, 4, 4, \ldots\ldots, 4\right\}$$

This equation yields

$$y_0 = 0 \qquad\qquad\qquad\qquad n = 0$$

$$y_1 = 0.25C/D \qquad\qquad\qquad n = 1$$

$$y_n = (Ay_{n-1} + By_{n-2} + 0.5C)/D \qquad 2 \leqslant n \leqslant 19$$

$$y_{20} = (Ay_{19} + By_{18} + 0.75C)/D \qquad n = 20$$

$$y_n = (Ay_{n-1} + By_{n-2} + C)/D \qquad\qquad n \geq 21$$

The graphical and numerical results are shown in Fig. 4 and Table 9 respectively. In Fig. 4, we can see that the transient response after the second step of compensation is better than the original Butterworth filter response. The maximum overshoot is reduced from 0.0431 to none but the delay time increases to 0.52 seconds.

### Compensation of Third Order Digital Butterworth Filter

A. First step of compensation

Table 6 gives m = 35

Thus

$$\bar{y} = (\frac{1}{s})(\frac{1+z^{35}}{2})(\frac{1}{s^3+2s^2+2s+1}) \qquad\qquad (47)$$

The block diagram of this system is shown in Fig. 7(c).
Calculation yields

$$z\bar{y}\left[z^3(T-1) + z^2(3-T-2T^2+0.5T^3) + z(0.5T^3+2T^2-T-3) + (T+1)\right]$$

$$= \frac{T^3 z(z^2+4z+1)(1+z^{35})}{12(1-z)} \qquad\qquad (48)$$

Then the recurrence relation is

$$y_n = \frac{(Ay_{n-1}+By_{n-2}+Cy_{n-3})}{R} + \frac{D}{12R}x_n \qquad\qquad (49)$$

$$x_n = \left\{0,\ 1,\ 5,\ 6,\ 6,\ \ldots\ldots,\ 6,\ 7,\ 11,\ 12,\ 12,\ \ldots\ldots,\ 12,\ \ldots\ldots\right\}$$

Where

$$A = 3+T-2T^2-0.5T^3$$

$$B = T+2T^2-0.5T^3-3$$

$$C = 1-T$$

$$D = T^3$$

$$R = 1+T$$

Table 9. $y(t) = \mathcal{L}^{-1}(\frac{1+z^{19}}{2} \cdot \frac{1/s}{1+ 2(0.427)s+(0.427)^2 s^2})$,
$y(0)=0$, $T=0.1$ sec.

| t | y(t) | t | y(t) | t | y(t) |
|-----|--------|-----|--------|-----|--------|
| 0.0 | 0.0000 | 2.7 | 0.8634 | 5.4 | 0.9987 |
| 0.1 | 0.0025 | 2.8 | 0.8851 | 5.5 | 0.9989 |
| 0.2 | 0.1047 | 2.9 | 0.9034 | 5.6 | 0.9991 |
| 0.3 | 0.1606 | 3.0 | 0.9188 | 5.7 | 0.9992 |
| 0.4 | 0.2166 | 3.1 | 0.9317 | 5.8 | 0.9994 |
| 0.5 | 0.2610 | 3.2 | 0.9425 | 5.9 | 0.9995 |
| 0.6 | 0.2992 | 3.3 | 0.9517 | 6.0 | 0.9995 |
| 0.7 | 0.3311 | 3.4 | 0.9594 | 6.1 | 0.9996 |
| 0.8 | 0.3579 | 3.5 | 0.9658 | 6.2 | 0.9997 |
| 0.9 | 0.3805 | 3.6 | 0.9712 | 6.3 | 0.9997 |
| 1.0 | 0.3995 | 3.7 | 0.9758 | 6.4 | 0.9998 |
| 1.1 | 0.4155 | 3.8 | 0.9797 | 6.5 | 0.9998 |
| 1.2 | 0.4289 | 3.9 | 0.9829 | 6.6 | 0.9999 |
| 1.3 | 0.4402 | 4.0 | 0.9856 | 6.7 | 0.9999 |
| 1.4 | 0.4497 | 4.1 | 0.9879 | 6.8 | 0.9999 |
| 1.5 | 0.4577 | 4.2 | 0.9898 | 6.9 | 0.9999 |
| 1.6 | 0.4644 | 4.3 | 0.9914 | 7.0 | 0.9999 |
| 1.7 | 0.4701 | 4.4 | 0.9928 | 7.1 | 0.9999 |
| 1.8 | 0.4748 | 4.5 | 0.9939 | 7.2 | 0.9999 |
| 1.9 | 0.4788 | 4.6 | 0.9949 | 7.3 | 1.0000 |
| 2.0 | 0.5339 | 4.7 | 0.9957 | 7.4 | 1.0000 |
| 2.1 | 0.6163 | 4.8 | 0.9964 | 7.5 | 1.0000 |
| 2.2 | 0.6748 | 4.9 | 0.9970 | 7.6 | 1.0000 |
| 2.3 | 0.7272 | 5.0 | 0.9975 | 7.7 | 1.0000 |
| 2.4 | 0.7704 | 5.1 | 0.9979 | 7.8 | 1.0000 |
| 2.5 | 0.8069 | 5.2 | 0.9982 | 7.9 | 1.0000 |
| 2.6 | 0.8376 | 5.3 | 0.9985 | 8.0 | 1.0000 |

Equation (49) yields

$$y_0 = 0 \qquad\qquad\qquad n = 0$$

$$y_1 = D/12R \qquad\qquad\qquad n = 1$$

$$y_2 = (Ay_1/R) + (5D/12R) \qquad\qquad n = 2$$

$$y_n = (Ay_{n-1} + By_{n-2} + Cy_{n-3} + 0.5D)/R \qquad 3 \leqslant n \leqslant 35$$

$$y_{36} = (Ay_{35} + By_{34} + Cy_{33})/R \; + 7R/12R \qquad n = 36$$

$$y_{37} = (Ay_{36} + By_{35} + Cy_{34})/R \; + 11D/12R \qquad n = 37$$

$$y_n = (Ay_{n-1} + By_{n-2} + Cy_{n-3} + D)/R \qquad\qquad n \geqslant 38$$

Results are presented in Table 10 and Fig. 5.

B. Second step of compensation

From Table 6 and Table 10, we get

$$T_{do} = 2.157 \text{ sec.}$$

$$T_{dl} = 3.75 \text{ sec.}$$

Then

$$\frac{T_{do}}{T_{dl}} = \frac{2.157}{3.75} = 0.573$$

Replacing s by 0.573s in the transfer function of equation (47) yields

$$\bar{y} = (\frac{1}{s})(\frac{1 + z^{20}}{2})(\frac{1}{0.188s^3 + 0.564s^2 + 1.146s + 1}) \qquad (50)$$

Applying the same process as in second order case yields

$$Z\bar{y} \left[ (0.188 + 0.327T) + (1.146T^2 - 0.327T - 0.564 + 0.5T^3)z \right.$$
$$\left. + (0.106 - 0.327T - 1.146T^2 + 0.5T^3)z^2 + (0.327T - 0.188)z^3 \right]$$

$$= \frac{T^3 z(z^2 + 4z + 1)(1 + z^{20})}{12(1-z)} \qquad (51)$$

Table 10. $y(t) = \mathcal{L}^{-1}\left(\dfrac{1+z^{35}}{2} \cdot \dfrac{1/s}{s^3+2s^2+2s+1}\right)$, $y(0)=0$, $T=0.1$ sec.

| t | y(t) | t | y(t) | t | y(t) |
|---|---|---|---|---|---|
| 0.0 | 0.0000 | 4.5 | 0.5853 | 9.0 | 1.0282 |
| 0.1 | 0.0000 | 4.6 | 0.6001 | 9.1 | 1.0267 |
| 0.2 | 0.0006 | 4.7 | 0.6157 | 9.2 | 1.0250 |
| 0.3 | 0.0019 | 4.8 | 0.6319 | 9.3 | 1.0232 |
| 0.4 | 0.0043 | 4.9 | 0.6488 | 9.4 | 1.0213 |
| 0.5 | 0.0080 | 5.0 | 0.6662 | 9.5 | 1.0194 |
| 0.6 | 0.0132 | 5.1 | 0.6840 | 9.6 | 1.0175 |
| 0.7 | 0.0199 | 5.2 | 0.7022 | 9.7 | 1.0156 |
| 0.8 | 0.0281 | 5.3 | 0.7206 | 9.8 | 1.0137 |
| 0.9 | 0.0379 | 5.4 | 0.7390 | 9.9 | 1.0118 |
| 1.0 | 0.0432 | 5.5 | 0.7575 | 10.0 | 1.0101 |
| 1.1 | 0.0621 | 5.6 | 0.7758 | 10.1 | 1.0083 |
| 1.2 | 0.0763 | 5.7 | 0.7939 | 10.2 | 1.0067 |
| 1.3 | 0.0917 | 5.8 | 0.8118 | 10.3 | 1.0051 |
| 1.4 | 0.1082 | 5.9 | 0.8292 | 10.4 | 1.0037 |
| 1.5 | 0.1258 | 6.0 | 0.8461 | 10.5 | 1.0023 |
| 1.6 | 0.1441 | 6.1 | 0.8625 | 10.6 | 1.0010 |
| 1.7 | 0.1632 | 6.2 | 0.8783 | 10.7 | 0.9999 |
| 1.8 | 0.1827 | 6.3 | 0.8934 | 10.8 | 0.9988 |
| 1.9 | 0.2027 | 6.4 | 0.9078 | 10.9 | 0.9979 |
| 2.0 | 0.2228 | 6.5 | 0.9214 | 11.0 | 0.9970 |
| 2.1 | 0.2430 | 6.6 | 0.9342 | 11.1 | 0.9963 |
| 2.2 | 0.2632 | 6.7 | 0.9462 | 11.2 | 0.9957 |
| 2.3 | 0.2832 | 6.8 | 0.9574 | 11.3 | 0.9952 |
| 2.4 | 0.3028 | 6.9 | 0.9677 | 11.4 | 0.9947 |
| 2.5 | 0.3220 | 7.0 | 0.9772 | 11.5 | 0.9944 |
| 2.6 | 0.3407 | 7.1 | 0.9859 | 11.6 | 0.9941 |
| 2.7 | 0.3588 | 7.2 | 0.9937 | 11.7 | 0.9939 |
| 2.8 | 0.3762 | 7.3 | 1.0007 | 11.8 | 0.9938 |
| 2.9 | 0.3928 | 7.4 | 1.0070 | 11.9 | 0.9938 |
| 3.0 | 0.4086 | 7.5 | 1.0125 | 12.0 | 0.9938 |
| 3.1 | 0.4235 | 7.6 | 1.0172 | 12.1 | 0.9939 |
| 3.2 | 0.4375 | 7.7 | 1.0212 | 12.2 | 0.9940 |
| 3.3 | 0.4506 | 7.8 | 1.0246 | 12.3 | 0.9942 |
| 3.4 | 0.4627 | 7.9 | 1.0274 | 12.4 | 0.9944 |
| 3.5 | 0.4739 | 8.0 | 1.0296 | 12.5 | 0.9946 |
| 3.6 | 0.4841 | 8.1 | 1.0312 | 12.6 | 0.9949 |
| 3.7 | 0.4939 | 8.2 | 1.0323 | 12.7 | 0.9952 |
| 3.8 | 0.5035 | 8.3 | 1.0330 | 12.8 | 0.9955 |
| 3.9 | 0.5134 | 8.4 | 1.0333 | 12.9 | 0.9958 |
| 4.0 | 0.5236 | 8.5 | 1.0331 | 13.0 | 0.9962 |
| 4.1 | 0.5344 | 8.6 | 1.0327 | 13.1 | 0.9965 |
| 4.2 | 0.5459 | 8.7 | 1.0319 | 13.2 | 0.9968 |
| 4.3 | 0.5582 | 8.8 | 1.0309 | 13.3 | 0.9972 |
| 4.4 | 0.5713 | 8.9 | 1.0297 | 13.4 | 0.9975 |

A: Transient response before compensa-
tion.

B: Transient response after first step
of compensation.

C: Transient response after second step
of compensation.

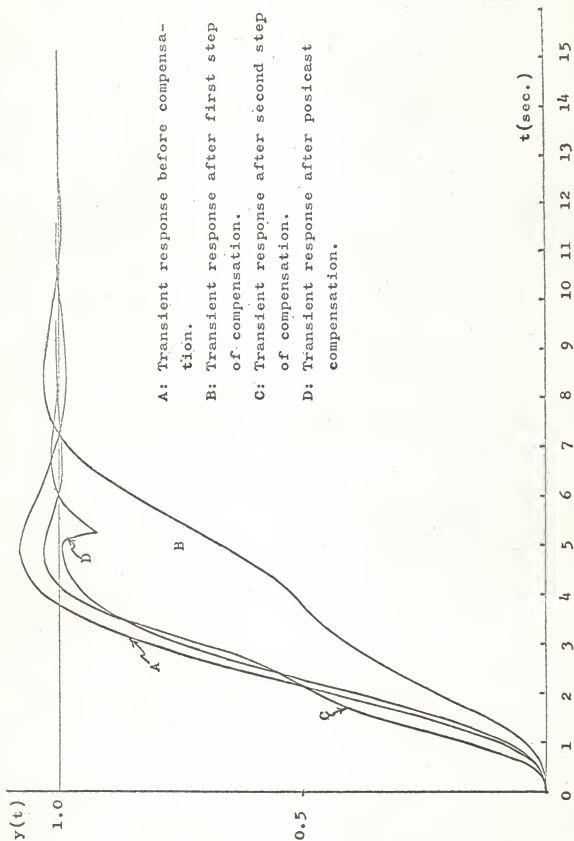D: Transient response after posicast
compensation.

Fig. 5. Compensation of third order digital Butterworth filter with unit step
function input.

Then the recurrence relation is

$$y_n = \frac{(Ay_{n-1} + By_{n-2} + Cy_{n-3})}{R} + \frac{DX_n}{12R} \qquad (52)$$

$$x_n = \left\{ 0,1,5,6,6,\ldots\ldots,6,7,11,12,12,\ldots\ldots12,\ldots\ldots \right\}$$

Where

$$A = 0.564 + 0.327T - 1.146T^2 - 0.5T^3$$

$$B = 0.327T + 1.146T^2 - 0.5T^3 - 0.564$$

$$C = 0.188 - 0.327T$$

$$D = T^3$$

$$R = 0.188 + 0.327T$$

Equation (52) yields

$$y_0 = 0 \qquad\qquad\qquad n = 0$$

$$y_1 = D/12R \qquad\qquad\qquad n = 1$$

$$y_2 = (Ay_{1/R}) + (5D/12R) \qquad\qquad n = 2$$

$$y_n = (Ay_{n-1} + By_{n-2} + Cy_{n-3} + 0.5D)/R \qquad 3 \leqslant n \leqslant 20$$

$$y_{21} = (Ay_{20} + By_{19} + Cy_{18})/R + (7D/12R) \qquad n = 21$$

$$y_{22} = (Ay_{21} + By_{20} + Cy_{19})/R + (11D/12R) \qquad n = 22$$

$$y_n = (Ay_{n-1} + By_{n-2} + Cy_{n-3} + D)/R \qquad n \geqslant 23$$

The graphical and numerical results are presented in Fig. 5 and Table 11 respectively. The maximum overshoot reduces from 0.0811 to 0.0328 and the time delay reduces from 2.157 seconds to 2.15 seconds.

Compensation of Fourth Order Digital Butterworth Filter

A. First step of compensation

Table 7 gives m = 33.

Table 11. $y(t) = \mathcal{L}^{-1}\left(\frac{1+z^{20}}{2} \cdot \frac{1}{0.188s^3+0.654s^2+1.146s+1}\right)$,

$y(0)=0$, $T=0.1$ sec.

| t | y(t) | t | y(t) | t | y(t) |
|---|------|---|------|---|------|
| 0.0 | 0.0000 | 4.2 | 1.0033 | 8.4 | 1.0005 |
| 0.1 | 0.0004 | 4.3 | 1.0131 | 8.5 | 1.0007 |
| 0.2 | 0.0029 | 4.4 | 1.0207 | 8.6 | 1.0008 |
| 0.3 | 0.0090 | 4.5 | 1.0262 | 8.7 | 1.0009 |
| 0.4 | 0.0196 | 4.6 | 1.0299 | 8.8 | 1.0009 |
| 0.5 | 0.0350 | 4.7 | 1.0320 | 8.9 | 1.0010 |
| 0.6 | 0.0552 | 4.8 | 1.0328 | 9.0 | 1.0009 |
| 0.7 | 0.0796 | 4.9 | 1.0324 | 9.1 | 1.0009 |
| 0.8 | 0.1078 | 5.0 | 1.0310 | 9.2 | 1.0008 |
| 0.9 | 0.1390 | 5.1 | 1.0289 | 9.3 | 1.0008 |
| 1.0 | 0.1725 | 5.2 | 1.0263 | 9.4 | 1.0007 |
| 1.1 | 0.2072 | 5.3 | 1.0233 | 9.5 | 1.0006 |
| 1.2 | 0.2426 | 5.4 | 1.0201 | 9.6 | 1.0005 |
| 1.3 | 0.2778 | 5.5 | 1.0168 | 9.7 | 1.0004 |
| 1.4 | 0.3120 | 5.6 | 1.0135 | 9.8 | 1.0003 |
| 1.5 | 0.3449 | 5.7 | 1.0104 | 9.9 | 1.0002 |
| 1.6 | 0.3757 | 5.8 | 1.0074 | 10.0 | 1.0002 |
| 1.7 | 0.4043 | 5.9 | 1.0047 | 10.1 | 1.0001 |
| 1.8 | 0.4301 | 6.0 | 1.0023 | 10.2 | 1.0000 |
| 1.9 | 0.4532 | 6.1 | 1.0002 | 10.3 | 1.0000 |
| 2.0 | 0.4733 | 6.2 | 0.9984 | 10.4 | 0.9999 |
| 2.1 | 0.4909 | 6.3 | 0.9969 | 10.5 | 0.9999 |
| 2.2 | 0.5078 | 6.4 | 0.9958 | 10.6 | 0.9999 |
| 2.3 | 0.5255 | 6.5 | 0.9949 | 10.7 | 0.9999 |
| 2.4 | 0.5451 | 6.6 | 0.9944 | 10.8 | 0.9998 |
| 2.5 | 0.5671 | 6.7 | 0.9940 | 10.9 | 0.9998 |
| 2.6 | 0.5917 | 6.8 | 0.9939 | 11.0 | 0.9998 |
| 2.7 | 0.6187 | 6.9 | 0.9940 | 11.1 | 0.9998 |
| 2.8 | 0.6478 | 7.0 | 0.9942 | 11.2 | 0.9999 |
| 2.9 | 0.6785 | 7.1 | 0.9946 | 11.3 | 0.9999 |
| 3.0 | 0.7103 | 7.2 | 0.9950 | 11.4 | 0.9999 |
| 3.1 | 0.7427 | 7.3 | 0.9956 | 11.5 | 0.9999 |
| 3.2 | 0.7749 | 7.4 | 0.9961 | 11.6 | 0.9999 |
| 3.3 | 0.8064 | 7.5 | 0.9967 | 11.7 | 0.9999 |
| 3.4 | 0.8369 | 7.6 | 0.9973 | 11.8 | 0.9999 |
| 3.5 | 0.8657 | 7.7 | 0.9978 | 11.9 | 1.0000 |
| 3.6 | 0.8926 | 7.8 | 0.9983 | 12.0 | 1.0000 |
| 3.7 | 0.9173 | 7.9 | 0.9988 | 12.1 | 1.0000 |
| 3.8 | 0.9396 | 8.0 | 0.9993 | 12.2 | 1.0000 |
| 3.9 | 0.9594 | 8.1 | 0.9997 | 12.3 | 1.0000 |
| 4.0 | 0.9766 | 8.2 | 1.0000 | 12.4 | 1.0000 |
| 4.1 | 0.9912 | 8.3 | 1.0003 | 12.5 | 1.0000 |

Thus

$$\bar{y} = (\frac{1}{s})(\frac{1+z^{33}}{2})(\frac{1}{s^4+2.6s^3+3.4s^2+2.6s+1}) \qquad (53)$$

The block diagram is shown in Fig. 7(d). Following the same procedures as in the previous cases yields

$$z\bar{y}\left[(1+1.3T)+z(-4-2.6T+3.4T^2+1.3T^3+W)+z^2(6-6.8T^2+4W)+ \right.$$
$$\left. z^3(-4+2.6T+3.4T^2-1.3T^3+W)+z^4(1-1.3T)\right]$$

$$= \frac{T^4z(z^3+11z^2+11z+1)(1+z^{33})}{48(1-z)} \qquad (54)$$

where $W=T^4/6$

Therefore the recurrence relation is

$$y_n = \frac{(Ay_{n-1}+By_{n-2}+Cy_{n-3}+Dy_{n-4})}{F} + (\frac{E}{F})x_n \qquad (55)$$

$$x_n = \left\{0, 1, 12, 23, 24, 24, \ldots\ldots, 24, 25, 36, 47,\right.$$
$$\left. 48, 48, \ldots\ldots\ldots\right\}$$

where

$$A = 4+2.6T+3.4T^2-1.3T^3-W$$
$$B = 6.8T^2-4W-6$$
$$C = 4-2.6T-3.4T^2+1.3T^3-W$$
$$D = 1.3T-1$$
$$E = \frac{T^4}{48}$$
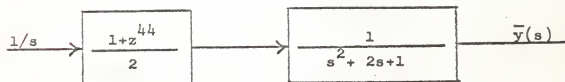$$F = 1+1.3T$$

Equation (55) yields

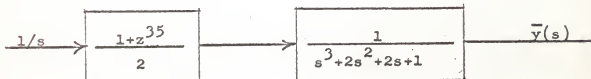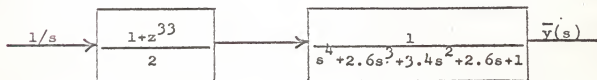| | |
|---|---|
| $y_0 = 0$ | $n = 0$ |
| $y_1 = E/F$ | $n = 1$ |
| $y_2 = (Ay_1+12E)/F$ | $n = 2$ |

(a)

(b)

(c)

(d)

Fig. 7. Digital Butterworth filters with pre-compensators

$$y_3 = \frac{Ay_2 + By_1 + 23E}{F} \qquad\qquad n = 3$$

$$y_n = \frac{Ay_{n-1} + By_{n-2} + Cy_{n-3} + Dy_{n-4} + 24E}{F} \qquad\qquad 4 \leqslant n \leqslant 33$$

$$y_{34} = \frac{Ay_{33} + By_{32} + Cy_{31} + Dy_{30} + 25E}{F} \qquad\qquad n = 34$$

$$y_{35} = \frac{Ay_{34} + By_{33} + Cy_{32} + Dy_{31} + 36E}{F} \qquad\qquad n = 35$$

$$y_{36} = \frac{Ay_{35} + By_{34} + Cy_{33} + Dy_{32} + 47E}{F} \qquad\qquad n = 36$$

$$y_n = \frac{Ay_{n-1} + By_{n-2} + Cy_{n-3} + Dy_{n-4} + 48E}{F} \qquad\qquad n \geqslant 37$$

Results are presented in Table 12 and Fig. 6.

B. Second step of compensation

From Table 7 and Table 12 it is found,

$$T_{do} = 2.85 \text{ sec.}$$
$$T_{d1} = 4.25 \text{ sec.}$$

Then

$$\frac{T_{do}}{T_{d1}} = \frac{2.85}{4.25} = 0.67$$

Replacing s by 0.67s in the transfer function of equation (53) yields

$$\bar{y} = (\frac{1}{s})(\frac{1 + z^{22}}{2})(\frac{1}{0.202s^4 + 0.782s^3 + 1.527s^2 + 1.742s + 1}) \qquad (56)$$

Applying the same process as in the previous cases yields

$$z\bar{y} \left[ (0.202 + 0.391\,T) + z(0.871\,T^3 + 1.527T^2 - 0.782T - 0.808 + W) \right.$$
$$+ z^2(1.212 - 3.045T^2 + 4W) + z^3(W - 0.871\,T^3 + 1.527T^2 + 0.782T - 0.808)$$
$$\left. z^4(0.202 - 0.391\,T) \right]$$

Table 12. $y(t) = \mathcal{L}^{-1}(\dfrac{1+z^{33}}{2} \cdot \dfrac{1/s}{s^4+2.6s^3+3.4s^2+2.6s+1})$,

$y(0)=0$, T=0.1 sec.

| t | y(t) | t | y(t) | t | y(t) |
|---|------|---|------|---|------|
| 0.0 | 0.0000 | 4.5 | 0.5336 | 9.0 | 1.0410 |
| 0.1 | 0.0000 | 4.6 | 0.5487 | 9.1 | 1.0403 |
| 0.2 | 0.0000 | 4.7 | 0.5639 | 9.2 | 1.0392 |
| 0.3 | 0.0001 | 4.8 | 0.5794 | 9.3 | 1.0378 |
| 0.4 | 0.0004 | 4.9 | 0.5951 | 9.4 | 1.0361 |
| 0.5 | 0.0010 | 5.0 | 0.6110 | 9.5 | 1.0342 |
| 0.6 | 0.0019 | 5.1 | 0.6271 | 9.6 | 1.0321 |
| 0.7 | 0.0034 | 5.2 | 0.6435 | 9.7 | 1.0298 |
| 0.8 | 0.0055 | 5.3 | 0.6601 | 9.8 | 1.0274 |
| 0.9 | 0.0084 | 5.4 | 0.6770 | 9.9 | 1.0250 |
| 1.0 | 0.0121 | 5.5 | 0.6939 | 10.0 | 1.0225 |
| 1.1 | 0.0167 | 5.6 | 0.7110 | 10.1 | 1.0199 |
| 1.2 | 0.0224 | 5.7 | 0.7282 | 10.2 | 1.0174 |
| 1.3 | 0.0291 | 5.8 | 0.7454 | 10.3 | 1.0149 |
| 1.4 | 0.0369 | 5.9 | 0.7625 | 10.4 | 1.0124 |
| 1.5 | 0.0458 | 6.0 | 0.7796 | 10.5 | 1.0100 |
| 1.6 | 0.0559 | 6.1 | 0.7965 | 10.6 | 1.0077 |
| 1.7 | 0.0671 | 6.2 | 0.8132 | 10.7 | 1.0055 |
| 1.8 | 0.0794 | 6.3 | 0.8296 | 10.8 | 1.0034 |
| 1.9 | 0.0928 | 6.4 | 0.8456 | 10.9 | 1.0015 |
| 2.0 | 0.1071 | 6.5 | 0.8613 | 11.0 | 0.9996 |
| 2.1 | 0.1224 | 6.6 | 0.8765 | 11.1 | 0.9980 |
| 2.2 | 0.1385 | 6.7 | 0.8912 | 11.2 | 0.9965 |
| 2.3 | 0.1553 | 6.8 | 0.9053 | 11.3 | 0.9951 |
| 2.4 | 0.1729 | 6.9 | 0.9189 | 11.4 | 0.9939 |
| 2.5 | 0.1909 | 7.0 | 0.9318 | 11.5 | 0.9928 |
| 2.6 | 0.2094 | 7.1 | 0.9440 | 11.6 | 0.9920 |
| 2.7 | 0.2283 | 7.2 | 0.9556 | 11.7 | 0.9912 |
| 2.8 | 0.2473 | 7.3 | 0.9664 | 11.8 | 0.9906 |
| 2.9 | 0.2665 | 7.4 | 0.9765 | 11.9 | 0.9902 |
| 3.0 | 0.2856 | 7.5 | 0.9585 | 12.0 | 0.9899 |
| 3.1 | 0.3047 | 7.6 | 0.9943 | 12.1 | 0.9897 |
| 3.2 | 0.3235 | 7.7 | 1.0021 | 12.2 | 0.9896 |
| 3.3 | 0.3419 | 7.8 | 1.0091 | 12.3 | 0.9896 |
| 3.4 | 0.3600 | 7.9 | 1.0154 | 12.4 | 0.9898 |
| 3.5 | 0.3776 | 8.0 | 1.0209 | 12.5 | 0.9900 |
| 3.6 | 0.3947 | 8.1 | 1.0257 | 12.6 | 0.9904 |
| 3.7 | 0.4113 | 8.2 | 1.0298 | 12.7 | 0.9908 |
| 3.8 | 0.4274 | 8.3 | 1.0332 | 12.8 | 0.9912 |
| 3.9 | 0.4431 | 8.4 | 1.0360 | 12.9 | 0.9917 |
| 4.0 | 0.4585 | 8.5 | 1.0382 | 13.0 | 0.9923 |
| 4.1 | 0.4737 | 8.6 | 1.0398 | 13.1 | 0.9929 |
| 4.2 | 0.4887 | 8.7 | 1.0408 | 13.2 | 0.9935 |
| 4.3 | 0.5037 | 8.8 | 1.0413 | 13.3 | 0.9942 |
| 4.4 | 0.5186 | 8.9 | 1.0414 | 13.4 | 0.9949 |

A: Transient response before compensation.
B: Transient response after first step of compensation.
C: Transient response after second step of compensation.
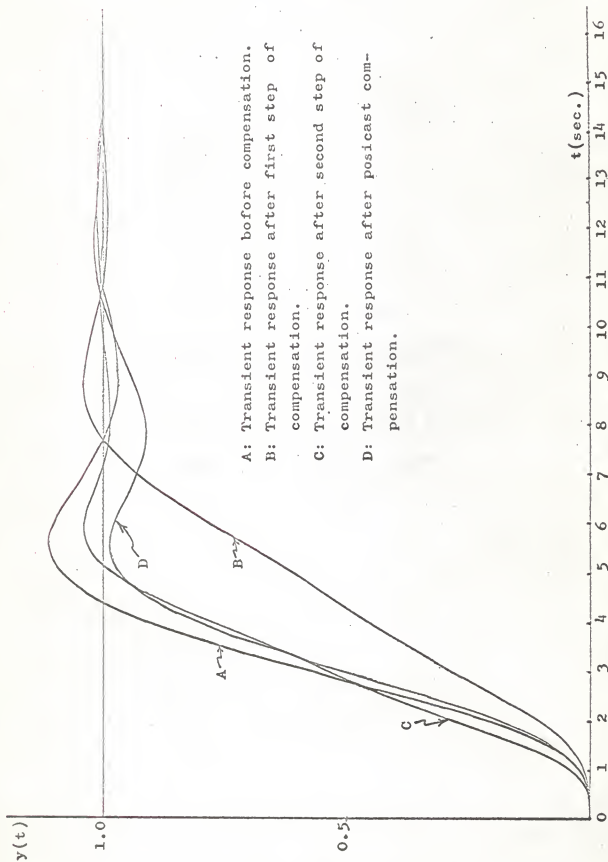D: Transient response after posicast compensation.

Fig. 6. Compensation of fourth order digital Butterworth filter with unit step function input.

$$= \frac{T^4 z(z^3 + 11z^2 + 11z + 1)(1 + z^{22})}{48(1-z)} \qquad (57)$$

The recurrence relation is

$$y_n = \frac{Ay_{n-1} + By_{n-2} + Cy_{n-3} + Dy_{n-4}}{F} + \left(\frac{E}{F}\right) x_n \qquad (58)$$

$$x_n = \left\{ 0, \ 1, \ 12, \ 23, \ 24, \ \ldots\ldots, \ 24, \ 25, \ 36, \ 47, \ 48, \ 48, \ \ldots \right.$$
$$\left. 48, \ \ldots\ldots \right\}$$

where
$$A = 0.808 + 0.782T - 1.527T^2 - 0.871T^3 - W$$

$$B = 3.054T^2 - 4W - 1.212$$

$$C = 0.808 - 0.782T - 1.527T^2 + 0.871T^3 - W$$

$$D = 0.391T - 0.202$$

$$E = T^4/48$$

$$W = T^4/6$$

$$F = 0.391T + 0.202$$

Equation (58) yields

$$y_0 = 0 \qquad\qquad n = 0$$

$$y_1 = \frac{E}{F} \qquad\qquad n = 1$$

$$y_2 = \frac{Ay_1 + 12E}{F} \qquad\qquad n = 2$$

$$y_3 = \frac{Ay_2 + By_1 + 23E}{F} \qquad\qquad n = 3$$

$$y_n = \frac{Ay_{n-1} + By_{n-2} + Cy_{n-3} + Dy_{n-4} + 24E}{F} \qquad\qquad 4 \leq n \leq 22$$

$$y_{23} = \frac{Ay_{22} + By_{21} + Cy_{20} + Dy_{19} + 25E}{F} \qquad\qquad n = 23$$

$$y_{24} = \frac{Ay_{23} + By_{22} + Cy_{21} + Dy_{20} + 36E}{F} \qquad\qquad n = 24$$

$$y_{25} = \frac{Ay_{24} + By_{23} + Cy_{22} + Dy_{21} + 47E}{F} \qquad n = 25$$

$$y_n = \frac{Ay_{n-1} + By_{n-2} + Cy_{n-3} + Dy_{n-4} + 48E}{F} \qquad n \geqslant 26$$

Results are presented in Table 13 and Fig. 6. The maximum overshoot reduces from 0.1118 to 0.0401 and the time delay reduces from 2.85 seconds to 2.83 seconds.

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Table 13.  $y(t)=\mathcal{L}^{-1}(\dfrac{(1 + z^{22})/2 \ (1/s)}{0.202s^4+0.782s^3+1.527s^2+1.742s+1}),y(0)=0,$

T=0.1 sec.

| t | y(t) | t | y(t) | t | y(t) |
|---|------|---|------|---|------|
| 0.0 | 0.0000 | 4.4 | 0.8740 | 8.8 | 0.9920 |
| 0.1 | 0.0000 | 4.5 | 0.8958 | 8.9 | 0.9930 |
| 0.2 | 0.0001 | 4.6 | 0.9164 | 9.0 | 0.9940 |
| 0.3 | 0.0006 | 4.7 | 0.9355 | 9.1 | 0.9950 |
| 0.4 | 0.0019 | 4.8 | 0.9531 | 9.2 | 0.9960 |
| 0.5 | 0.0043 | 4.9 | 0.9691 | 9.3 | 0.9969 |
| 0.6 | 0.0082 | 5.0 | 0.9834 | 9.4 | 0.9978 |
| 0.7 | 0.0140 | 5.1 | 0.9960 | 9.5 | 0.9986 |
| 0.8 | 0.0219 | 5.2 | 1.0069 | 9.6 | 0.9994 |
| 0.9 | 0.0323 | 5.3 | 1.0161 | 9.7 | 1.0001 |
| 1.0 | 0.0451 | 5.4 | 1.0237 | 9.8 | 1.0006 |
| 1.1 | 0.0604 | 5.5 | 1.0297 | 9.9 | 1.0011 |
| 1.2 | 0.0782 | 5.6 | 1.0343 | 10.0 | 1.0015 |
| 1.3 | 0.0984 | 5.7 | 1.0374 | 10.1 | 1.0018 |
| 1.4 | 0.1207 | 5.8 | 1.0393 | 10.2 | 1.0020 |
| 1.5 | 0.1450 | 5.9 | 1.0401 | 10.3 | 1.0021 |
| 1.6 | 0.1708 | 6.0 | 1.0398 | 10.4 | 1.0022 |
| 1.7 | 0.1978 | 6.1 | 1.0387 | 10.5 | 1.0022 |
| 1.8 | 0.2258 | 6.2 | 1.0368 | 10.6 | 1.0021 |
| 1.9 | 0.2542 | 6.3 | 1.0342 | 10.7 | 1.0020 |
| 2.0 | 0.2828 | 6.4 | 1.0312 | 10.8 | 1.0018 |
| 2.1 | 0.3112 | 6.5 | 1.0279 | 10.9 | 1.0016 |
| 2.2 | 0.3390 | 6.6 | 1.0243 | 11.0 | 1.0013 |
| 2.3 | 0.3658 | 6.7 | 1.0205 | 11.1 | 1.0011 |
| 2.4 | 0.3916 | 6.8 | 1.0167 | 11.2 | 1.0008 |
| 2.5 | 0.4164 | 6.9 | 1.0130 | 11.3 | 1.0005 |
| 2.6 | 0.4402 | 7.0 | 1.0094 | 11.4 | 1.0002 |
| 2.7 | 0.4633 | 7.1 | 1.0059 | 11.5 | 1.0000 |
| 2.8 | 0.4860 | 7.2 | 1.0027 | 11.6 | 0.9997 |
| 2.9 | 0.5084 | 7.3 | 0.9998 | 11.7 | 0.9995 |
| 3.0 | 0.5310 | 7.4 | 0.9972 | 11.8 | 0.9992 |
| 3.1 | 0.5537 | 7.5 | 0.9949 | 11.9 | 0.9990 |
| 3.2 | 0.5769 | 7.6 | 0.9930 | 12.0 | 0.9989 |
| 3.3 | 0.6006 | 7.7 | 0.9914 | 12.1 | 0.9987 |
| 3.4 | 0.6248 | 7.8 | 0.9902 | 12.2 | 0.9986 |
| 3.5 | 0.6495 | 7.9 | 0.9893 | 12.3 | 0.9985 |
| 3.6 | 0.6746 | 8.0 | 0.9887 | 12.4 | 0.9984 |
| 3.7 | 0.7001 | 8.1 | 0.9884 | 12.5 | 0.9984 |
| 3.8 | 0.7259 | 8.2 | 0.9883 | 12.6 | 0.9983 |
| 3.9 | 0.7516 | 8.3 | 0.9885 | 12.7 | 0.9983 |
| 4.0 | 0.7772 | 8.4 | 0.9889 | 12.8 | 0.9984 |
| 4.1 | 0.8024 | 8.5 | 0.9895 | 12.9 | 0.9984 |
| 4.2 | 0.8271 | 8.6 | 0.9902 | 13.0 | 0.9985 |
| 4.3 | 0.8510 | 8.7 | 0.9911 | 13.1 | 0.9985 |

## DISCUSSION

Investigation of the graphs and numerical results in the uncompensated cases shows that as the order of the Butterworth filter increases the ratio of rise time to half-cycle overshoot duration, $mT$, increases; this behavior is shown in Fig. 8. Thus digital filter compensation is most effective on higher order Butterworth filters and a smoother transient response curve results as seen in Fig.'s 4, 5, 6.

The transient response of the higher order Butterworth filter with unit step input always has larger maximum overshoot and time delay which cannot be extensively reduced, the first time when the digital filter compensation is introduced. However, the process can be repeated until the desired reduction in transient overshoot is obtained; this is the goal of digital filter compensation.

For comparison, posicast control compensation is applied to each case and the resulting transient responses are presented in Fig.'s 4, 5, 6. In the second order case, the two compensating methods give almost the same result, although there is inflection in the digital filter compensation case. In the third order case large undershoot is produced with posicast compensation and digital filter compensation gives better results. In the fourth order case, oscillation with much larger undershoot occurs under posicast control compensation; however, digital filter compensation results in much better behavior.
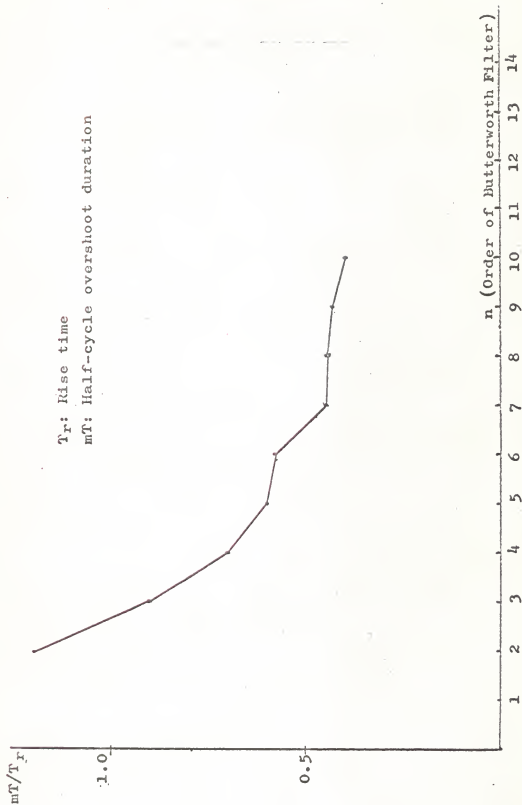
**Fig. 8.** Behavior of the ratio of rise time to half-cycle overshoot duration in the transient responses of Butterworth filters with unit step function inputs.

$T_r$: Rise time

$mT$: Half-cycle overshoot duration

n (Order of Butterworth Filter)

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REFERENCES

1. Criswell, M. L.
   An algorithm for computing the Z-transform of $t^n/n!$. ITM-79,
   Bendix Systems Division, The Bendix Corporation, Ann Arbor,
   Michigan, July, 1962.

2. Golay, Marcel J. E.
   Polynomials of transfer functions. IRE Transactions on Circuit
   Theory, 3:224-229. September, 1960.

3. Halijak, C. A.
   Digital approximation of differential equations by trapezoidal
   convolution. Proceedings of the Central States Simulation Council
   Meeting of Extrapolation of Analog Computation Methods, Kansas
   Engineering Experiment Station, Special Report No. 10, Vol. 45,
   No. 7, Kansas State University, Manhattan, Kansas, July 1961,
   pp. 83-94.

4. Naumov, B.
   Approximate method for calculating the time response in linear,
   time-varying and nonlinear automatic control systems. Trans. ASME,
   Journal of Basic Engineering, Vol. 83, Series D, No. 1, pp. 109-118,
   March, 1961.

5. Pullen, K. A., Jr.
   A personal letter to Dr. C. A. Halijak.

6. Smith, Otto J. M.
   Feedback control systems. New York and London: McGraw-Hill, 1958.
   pp. 331-341.

7. Van Valkenburg, M. E.
   Introduction to modern network synthesis. New York: John Wiley
   and Sons, 1960. p. 376.

DIGITAL FILTERS

by

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AN ABSTRACT OF A MASTER'S REPORT

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The essential content of this report is to analyze Butterworth filters in the time domain by using a digital method for finding their transient performances. Digital filter compensation is effected by multiplication in the Z-transform domain by $(1+z^m)/2$. For comparison, posicast compensation of a compensator $(1-P)+Pz^m$ is employed in each digital filter compensation case. The approximate digital analysis is accomplished by the Naumov program which is an improved accuracy trapezoidal convolution program.

Transient responses of higher order digital Butterworth filters employing digital filter compensation are much better than those employing posicast compensation. Digital filter compensation can be repeated until desired reduction in transient overshoot is obtained--the goal of digital filter compensation.

This report introduces an interesting subject, the pseudo-Butterworth function. Pseudo-Butterworth functions can be converted into approximate Butterworth functions by replacing $s$ with $s/(b_n)^{1/n}$, where $b_n$ is the coefficient of the $n^{th}$ term of pseudo-Butterworth polynomial of order $n$.